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Unstable State Dynamics: Treatment of Colored Noise

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The diffusion of a particle set near an unstable point in a bistable potential is considered. The scaling theory of fluctuations proposed originally for one-dimensional systems driven by Gaussian white noise is extended to arbitrary dimensions. The merits and drawbacks of the scaling theory are discussed by taking a model problem in one dimension. It is shown in passing that the saddle point approximation enables one to get analytic expressions for various moments of the stochastic process. The two different methods to include asymptotic fluctuations—which are absent in the usual scaling solution—are shown to be equivalent. An alternate way of including asymptotic fluctuations is attempted by solving the associated Fokker–Planck equation using the Fer formula. The reason for the failure of this method is traced. After this, it is argued that the unified scaling theory should be applicable for treatment of colored noise as well, for the scaling assumption is independent of the statistical property of the driving noise. Explicit Monte Carlo simulation of a model one-dimensional system driven by exponentially correlated Gaussian noise is performed and compared with the scaling solution to bolster this point. The agreement is very good.

KEY WORDS: Diffusion; Master equation; Fokker–Planck equation; system size expansion; statistical linearization; generalized linearization; scaling theory; noise, Fer formula; saddle point approximation; Monte Carlo simulation.

1. INTRODUCTION

The relaxation to equilibrium of physical systems (with a large number of degrees of freedom) far from equilibrium can often be described in terms of a few collective variables. Fluctuations, which arise from the large number of irrelevant degrees of freedom which are neglected in this statistical treatment, are an inevitable part of the measurement of these macroscopic quantities. The effect of the fluctuations are usually included[1–3] by adding

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Gaussian white noise terms to the evolution equations of these variables. Closed form solutions of these nonlinear Langevin equations are seldom possible and one resorts to approximations.

Fluctuations in physical systems are small[4] and that forms the basis for all the approximation methods for the solution of the nonlinear Langevin equations. Depending on the type of nonlinearity (nature of the potential) and the initial condition, three different scenarios of fluctuations are possible. Each of these has to be handled by a method devised specifically for that.

The simplest case arises when the system evolves to a single, globally stable, steady state. Then the fluctuations in the macroscopic quantity also remain small for all times. The average behavior of the system can be studied by the deterministic laws. The effect of fluctuations can be obtained well by using the system size expansion,[4] the statistical linearization,[5,6] and the generalized linearization schemes.[7–9]

The next scenario obtains when the system is near a critical point. The scaling up of fluctuations in the macroscopic quantity and the critical slowing down are the important features of this problem. This case has been handled satisfactorily by Dekker.[10] He splits the problem into an irreducible part and a corrective remainder.

In this paper, we will be concerned with the last case where a system in a bistable potential is suddenly set at the unstable equillbrium point. Fluctuations in the mcroscopic variable are small initially. There is a fluctuation enhancement in the intermediate time domain which is responsible for formation of macroscopic order. Asymptotically, the fluctuations regress to the equilbrium values around each of the stable steady states. There have been several more or less equivalent methods[11–20] to handle this situation. All these attempts, which are collectively referred to as scaling theory, have been restricted to one-dimensional systems. One of the objectives of the present work is to lift this limitation. We do this by elaborating on the method of nonlinear transformation due to de Pasquale and Tombesi.[14]

The scaling theory is successful in giving a fairly correct description of fluctuations in the initial and the intermdiate time regimes. The asymptotic fluctuations cannot be handled in the ambit of scaling theory.[9,12,19,21] There have been attempts by Suzuki[12] and Dekker[19] to include the asymptotic fluctuations. We have demonstrated the equivalence of these two seemingly different mthods. We have attempted to include the asymptotic fluctuations by solving the Fokker–Planck equation, associated with the Langevin equation, using the Fer formula.[22] This solution, though slightly better than the scaling result for the initial and the intermdiate times, suffers from the same drawback at large times. The

reason for this failure is traced to be the noninvertibility of the nonlinear transformation. It is then argued that Suzuki's unified treatment[12] (equivalent to Dekker's systematic evaluation of the Master equation[19]) is essential for restoring the asymptotic fluctuations, even if we can obtain the exact probability distribution in the transformed variable.

As mentioned in the beginning, the intrinsic fluctuations are often modeled as Gaussian white noise. Though this allows considerable simplification of the mathematical analysis, the real physical situation may be far from this—the intrinsic fluctuations will most probably have a finite correlation time.[23] Further, it is not always necessary that the fluctuations respect a Gaussian distribution. The second objective of the present paper is to argue that the scaling method is valid for any distribution of the driving noise. This is based on the fact that the scaling approximation is independent of the statistical property of the driving noise. Monte Carlo simulation of a model one-dimensional system driven by exponentially correlated Gaussian noise shows remarkable agreement with the scaling result.

The plan of the paper is as follows: The nonlinear transformation of the stochastic process in arbitrary dimensions and the scaling solution applicable for any distribution of the driving noise is presented in Section 2. Section 3 is devoted to a detailed study of the Gaussian white noise problem. Systems driven by exponentially correlated Gaussian noise is dealt with in Section 4. The conclusions are brought out in Section 5.

2. THE NONLINEAR TRANSFORMATION AND THE SCALING SOLUTION[2]

We consider the diffusion of a particle in a bistable potential, in an N-dimensional space $(X_1,..., X_N)$, modeled by the stochastic differential equations

$$\dot{X}_i = C_i(\bar{X}) + f_i(t) \tag{2.1}$$

In Eq. (2.1) the overdot refers to differentiation with respect to time t, $C_i(\bar{X})$ denote the drift terms and $f_i(t)$ represent the driving noise terms. Let $\bar{a}$ to be a representative point on the unstable surface on which the particle will be set suddenly at time $t=0$. Then by definition of instability, $C_i(\bar{a})=0$; $\tilde{\Gamma}$ positive definite, where

$$\tilde{\Gamma}_{ij} = \left.\frac{\partial C_i}{\partial X_j}\right|_{\bar{X}=\bar{a}} \tag{2.2}$$

[2] Through out this paper, an overbar on a quantity indicates that it is a vector. Expectation values are indicated by the angular brackets $\langle\ \rangle$.

We then outline the approximate solution of Eq. (2.1), with the initial condition $P_{\bar{X}}(\bar{X}, 0) = \delta(\bar{X} - \bar{a})$, in the following four steps:

Step 1. Determine the time-dependent invariants $Z_i = Z_i(t, \bar{Y})$ of the corresponding deterministic system

$$\dot{Y}_i = C_i(\bar{Y}), \qquad Y_i(0) = Z_i \tag{2.3}$$

By integrating Eq. (2.3) formally and inverting for Z_i, the time-dependent invariants can be obtained as

$$Z_i = \exp\left[-t \sum_{j=1}^{N} C_j(\bar{Y}) \frac{\partial}{\partial Y_j} \right] Y_i \tag{2.4}$$

Step 2. Instead of the original stochastic process $\bar{X}(t)$ introduce a new stochastic process $\bar{\xi}(t)$ such that $\bar{\xi}(t)$ reduces to the invariants $\bar{Z}$ in the limit of vanishing noise. Thus the $\bar{\xi}$ can be defined as

$$\xi_i = \exp\left[-t \sum_{j=1}^{N} C_j(\bar{X}) \frac{\partial}{\partial X_j} \right] X_i \tag{2.5}$$

It is easy to show that the $\bar{\xi}$ evolve as

$$\frac{d\xi_i}{dt} = \sum_{j=1}^{N} \frac{\partial \xi_i}{\partial X_j} f_j \tag{2.6}$$

Equation (2.6) is completely equivalent to Eq. (2.1). Its exact solution is as formidable as that of the latter.

Step 3. Apply scaling approximation to Eq. (2.6).

When the system evolves from an unstable point fluctuations are all important in the initial time regime. But, fluctuations are relatively unimportant and the nonlinearity is the most dominant one in the intermediate time region where the fluctuation enhancement occurs. By the nonlinear transformation given by Eq. (2.5) we have completely respected the nonlinearity. The initial fluctuations can correctly be incorporated by replacing $\partial \xi_i / \partial X_j$ in Eq. (2.6) by its value at the unstable point. Thus the scaling approximation is

$$\frac{\partial \xi_i^{(\mathrm{sc})}}{\partial X_j} = \left. \frac{\partial \xi_i}{\partial X_j} \right|_{\bar{X} = \bar{a}} \tag{2.7}$$

where the superscript denotes that the quantity is not the exact one but its approximate value in the scaling sense. Using Eqs. (2.5) and (2.2) it is easy to show that

$$\left.\frac{\partial \xi_i}{\partial X_j}\right|_{\bar{X}=\bar{a}} = (\exp[-\tilde{\Gamma} t])_{ij} \tag{2.8}$$

Thus the approximate solution of Eq. (2.1) can be obtained using

$$\frac{d\xi_i^{(\mathrm{sc})}}{dt} = \sum_{j=1}^{N} (\exp[-\tilde{\Gamma} t])_{ij} f_j \tag{2.9}$$

We stress that in arriving at Eq. (2.9) we have not assumed any particular statistical property of $\bar{f}$. The only assumption that has gone in is that the strength of $\bar{f}$ is small.

Equation (2.9) being linear, we can trivially integrate them to get $\bar{\xi}^{(\mathrm{sc})}(t)$ and its probability distribution function $P_{\bar{\xi}^{(\mathrm{sc})}}(\bar{\xi}^{(\mathrm{sc})}, t)$ as

$$\xi_j^{(\mathrm{sc})}(t) = a_i + g_i(t), \qquad g_i(t) = \sum_{j=1}^{N} \int_0^t (\exp[-\tilde{\Gamma} t_1])_{ij} f_j(t_1)\, dt_1 \tag{2.10}$$

and

$$P_{\bar{\xi}^{(\mathrm{sc})}}(\bar{\xi}^{(\mathrm{sc})}, t) = \mathrm{Prob}(\bar{a} + \bar{g}) \tag{2.11}$$

It is easy to get the corresponding results for the original variable $\bar{X}$. Inverting Eq. (2.10) formally using the functional equation (2.5), we get

$$X_i^{(\mathrm{sc})}(t) = Y_i(t, \bar{h}(t)), \qquad h_i(t) = a_i + g_i(t) \tag{2.12}$$

and

$$P_{\bar{X}^{(\mathrm{sc})}}(\bar{X}^{(\mathrm{sc})}, t) = \langle \delta(\bar{X}^{(\mathrm{sc})} - \bar{Y}(t, \bar{h}(t)) \rangle_{\bar{h}(t)} \tag{2.13}$$

Equation (2.12) means that the scaling solution is obtained from solution $\bar{Y}(t, \bar{Y}(t=0))$ of the corresponding deterministic equation by replacing the initial value $\bar{Y}(t=0) = \bar{a}$ by $\bar{a} + \bar{g}(t)$. If for a moment we assume that $\bar{g}(t)$ equilibrates fast, crudely speaking, the scaling method is equivalent to solving the deterministic equations with an appropriate stochastic initial condition. Rigorously, the initial condition itself changes at every instant. Hence the method, in practice, is equivalent to modeling the noise as a series of δ function pulses whose strength varies in time. To summarize, if the solution of the deterministic system is obtainable, then the scaling solution is known analytically.

Step 4. Obtain the systematic correction $\bar{R}$ to be added to the scaling solution $\bar{X}^{(\mathrm{sc})}$.

Following the unified scaling theory of Suzuki,[12] we express the original stochastic process $\bar{X}$ as

$$\bar{X} = \bar{X}^{(sc)} + \bar{R}(t) \tag{2.14}$$

Since the deterministic evolution subject to initial fluctuations is contained in $\bar{X}^{(sc)}$, $\bar{R}(t)$ should remain small for all times. Hence a simple linearization of the evolution equation of $\bar{R}$ should suffice. Using Eqs. (2.1) and (2.12) in conjunction with Eq. (2.3) we get (to the linear order)

$$\dot{R}_i = \sum_{j=1}^{N} \left.\frac{\partial C_i}{\partial X_j}\right|_{\bar{X}=\bar{X}^{(sc)}} R_j + f_i - \sum_{j,k=1}^{N} Y_{i,j}(\exp[-\tilde{\Gamma}t])_{jk} f_k \tag{2.15}$$

In Eq. (2.15), $Y_{i,j}$ stands for $(\partial/\partial h_j)\, Y_i(t, \bar{h})$. Since the asymptotic solution, *whenever it exists*, in independent of the initial state, $Y_{i,j}$ tends to zero asymptotically. Hence for reasonably large times, we may neglect the last term in Eq. (2.15).[3] To simplify the matter further, we may replace $(\partial C_i/\partial X_j)|_{\bar{X}=\bar{X}^{(sc)}}$ by its average value. The resulting equation for R reads

$$\dot{R}_i = \sum_{j=1}^{N} \left\langle \left.\frac{\partial C_i}{\partial X_j}\right|_{\bar{X}=\bar{X}^{(sc)}} \right\rangle R_{j+f_i} \tag{2.16}$$

Let $P_{\bar{R}}(\bar{R}, t)$ be the probability distribution of the correction terms. Assuming statistical independence of $\bar{X}^{(sc)}$ and $\bar{R}$, the probability distribution of $\bar{X}$ in the unified scaling theory can be written as

$$P_{\bar{X}}(\bar{X}, t) = \int P_{\bar{X}^{(sc)}}(\bar{u}, t)\, P_{\bar{R}}(\bar{X} - \bar{u}, t)\, du \tag{2.17}$$

Examples of the Nonlinear Transformation

It is possible to get closed form algebraic expression for the transformed variable in one dimension. Starting from the operator equation

$$\xi = \exp\left[-tC(X)\frac{\partial}{\partial X}\right] X \tag{2.18a}$$

It is trivial to show that

$$\xi = F^{-1}(e^{-\gamma t} F(X)) \tag{2.18b}$$

[3] For the case of linear $C(X)$ this is not true. The $\tilde{\Gamma}$ should be *negative definite* for the asymptotic solution to exist. Then, the $\exp[-\tilde{\Gamma}t]$ *increases* exactly at the same rate as the decease of $Y_{i,j}$ that the last term of Eq. (2.15) becomes equal to $-f_i$. Thus we get $R_i = 0$, as they should. However, for the problem of diffusion in a bistable potential, $\tilde{\Gamma}$ is positive definite. Hence the above approximation is justified in this context.

where

$$\gamma = \left.\frac{\partial C}{\partial X}\right|_{X=a}, \qquad F(X) = \exp\left[\int^X \frac{\gamma du}{C(u)}\right] \tag{2.18c}$$

For the specific case of $C(X) = \gamma X - gX^{2n+1}$, which will be used later,

$$\xi = Xe^{-\gamma t}\left[1 - \frac{g}{\gamma} X^{2n}(1 - e^{-2n\gamma t})\right]^{-1/2n} \tag{2.19}$$

It is not possible to derive such general formulas for arbitrary dimensions. Just for illustrating the transformation we consider a two-variable problem of relevance to laser physics

$$C_1(X_1, X_2) = \alpha X_1[1 - (X_1^2 + X_2^2)^n], \qquad C_2(X_1, X_2) = \alpha X_2[1 - (X_1^2 + X_2^2)^n] \tag{2.20a}$$

For these drift terms, the transformed variables are

$$\begin{aligned} \xi_1 &= e^{-\alpha t} X_1[1 - (X_1^2 + X_2^2)^n(1 - e^{-2n\alpha t})]^{-1/2n} \\ \xi_2 &= e^{-\alpha t} X_2[1 - (X_1^2 + X_2^2)^n(1 - e^{-2n\alpha t})]^{-1/2n} \end{aligned} \tag{2.20b}$$

We repeat that if the deterministic system is solvable, then the transformed variables also can be obtained in closed form. Even if $\bar{\xi}$ cannot be explicitly found, still $\bar{\xi}$ corresponding to any $\bar{X}$ may be numerically evaluated using Trotter's formula.[24–26]

3. TREATMENT OF GAUSSIAN WHITE NOISE

This section is devoted to a threadbare study of Gaussian white noise. For clarity of presentation, we restrict ourselves to a model system in one dimension

$$\dot{X} = \gamma X - gX^3 + f(t) \tag{3.1a}$$

and

$$\langle f(t) \rangle = 0, \qquad \langle f(t) f(t') \rangle = 2\varepsilon\, \delta(t - t') \tag{3.1b}$$

Then from Eq. (2.11), the scaling solution can be obtained as

$$P_{\xi^{(sc)}}(\xi, t) = \frac{1}{[2\pi(\varepsilon/\gamma)(1 - e^{-2\gamma t})]^{1/2}} \exp\left[\frac{-\xi^2}{(2\varepsilon/\gamma)(1 - e^{-2\gamma t})}\right] \tag{3.2a}$$

with

$$\xi = Xe^{-\gamma t}\left[1 - \frac{g}{\gamma}X^2(1-e^{-2\gamma t})\right]^{-1/2}, \qquad X = \xi e^{\gamma t}\left[1 + \frac{g}{\gamma}\xi^2(e^{2\gamma t}-1\right]^{-1/2} \tag{3.2b}$$

The moments of the stochastic process are obtained by a quadrature

$$\langle X^{2m+1}\rangle^{(sc)} = 0$$

$$\langle X^{2m}\rangle^{(sc)} = \frac{(\gamma/g)^m}{(1-e^{-2\gamma t})^m}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\left(\frac{v^2\tau}{1+v^2\tau}\right)^m e^{-v^2}\,dv \tag{3.3a}$$

where the scaling variable τ is given by

$$\tau = \frac{2\varepsilon g}{\gamma^2}(e^{2\gamma t}-1) \tag{3.3b}$$

Monte Carlo simulation of Eq. (3.1) was performed using Runge–Kutta–Gill method described earlier,[7,9] for the choice of parameters $\gamma = g = 1$ and $\varepsilon = 0.5 \times 10^{-6}$. The second and fourth moments are presented as a function of time in Figs. 1 and 2, respectively. The corresponding scal-

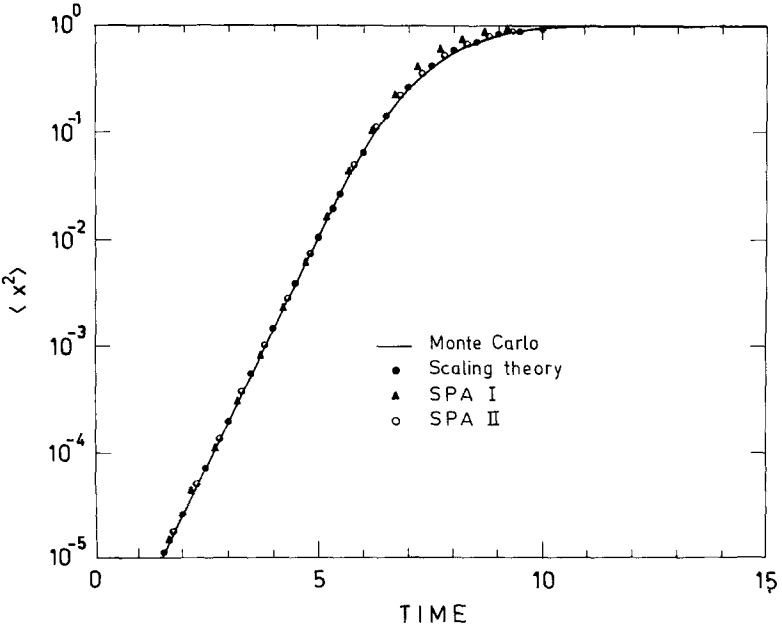

Fig. 1. The second moment $\langle X^2\rangle$, as a function of time of the stochastic process $\dot{X} = X - X^3 + f(t)$ with $P_X(X,0) = \delta(X)$. The $f(t)$ is taken to be Gaussian and white with $\langle f(t)\rangle = 0$ and $\langle f(t)f(t')\rangle = 10^{-6}\delta(t-t')$. The results obtained from the Monte Carlo simulation, the scaling theory, and the saddle point approximations (SPA I and SPA II) to that are presented.

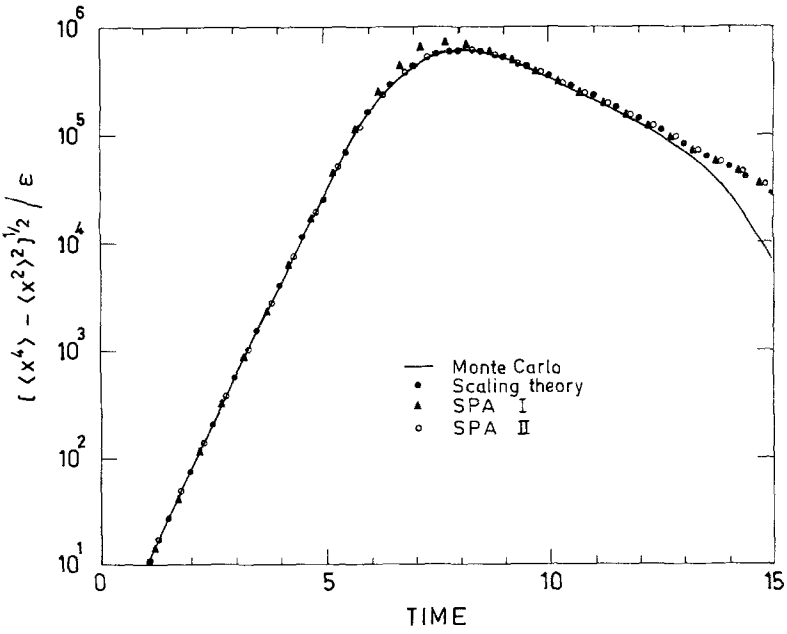

Fig. 2. The normalized standard deviation $(\langle X^4\rangle - \langle X^2\rangle^2)^{1/2}/\varepsilon$ of the process X^2 for the problem considered in Fig. 1.

ing results are also furnished. The comparison is seen to be very good in the initial and scaling regimes.

Since the integrand in Eq. (3.3) is of the form e^{-v^2}, we though it worthwhile to mention that the saddle point approximation[27] to the integral should be good. If we express $v^2\tau/(1+v^2\tau)$ as $1-1/(1+v^2\tau)$ and then perform the saddle point approximation, we obtain

$$\langle X^{2m}\rangle^{(\mathrm{sc})} \simeq \frac{(\gamma/g)^m}{[1-e^{-2\gamma t}]^m} \sum_{n=0}^{m} {}^mC_n(-1)^n[1+n\tau]^{-1/2} \tag{3.4}$$

On the other hand, if we perform the approximation directly on Eq. (3.3) we get

$$\langle X^{2m}\rangle^{(\mathrm{sc})} \simeq \frac{(\gamma/g)^m}{(1-e^{-2\gamma t})^m}\left[\frac{(1+4m\tau)^{1/2}-1}{(1+4m\tau)^{1/2}+1}\right]^m \times \left[\frac{(1+4m\tau)^{1/2}+1}{(1+4m\tau)^{1/2}}\right]^{1/2} \exp\left[-\frac{(1+4m\tau)^{1/2}-1}{2\tau}\right] \tag{3.5}$$

These results are also presented in Figs. 1 and 2. Both the methods are good at small and large times. The second one is better in the intermediate

region. The first one has the advantage that it allows even closed form expressions for such quantities as the splitting time.[12]

3.1. Incorporation of Asymptotic Fluctuations

It has been demonstrated that the scaling solution is very good for times $\sim \ln(1/\varepsilon)$. The asymptotic results are, however, not good. It is easy to show from Eq. (3.2) that the asymptotic distribution in $X^{(sc)}$ goes as

$$P_{X^{(sc)}}(X, \infty) = \delta(X^2 - \gamma/g) \tag{3.6}$$

That is, the probability distribution in X asymptotically degenerates into two δ functions centered around the stable steady states $\pm(\gamma/g)^{1/2}$.

There have been attempts by Suzuki[22] and Dekker[19] to include the asymptotic fluctuations. The equivalence of these two methods will be demonstrated in the remaining part of this section.

Suzuki's unified treatment of splitting X as the sum of the scaling solution $X^{(sc)}$ and correction $R(t)$ has already been discussed in Section 2. Using Eq. (2.16), we get

$$P_R(R, t) = \frac{1}{[2\pi\sigma_R^2(t)]^{1/2}} \exp\left[- \frac{R^2}{2\sigma_R^2(t)} \right] \tag{3.7}$$

where

$$\dot{\sigma}_R^2(t) = 2 \left\langle \left. \frac{\partial C}{\partial X} \right|_{X = X^{(sc)}} \right\rangle \sigma_R^2(t) + 2\varepsilon \tag{3.8}$$

Then the unified scaling solution is given by

$$P_X(X, t) = \int_{-\infty}^{\infty} du\, P_{X^{(sc)}}(u, t) \frac{1}{[2\pi\sigma_R^2(t)]^{1/2}} \exp\left[\frac{-(X-u)^2}{2\sigma_R^2(t)} \right] \tag{3.9}$$

From the property of Hermite polynomials,[27] it follows that

$$\langle X^n \rangle_{P_X} = (-i)^n \left(\frac{\sigma_R^2}{2} \right)^{n/2} \left\langle H_n\left(\frac{iX}{(2\sigma_R^2)^{1/2}} \right) \right\rangle P_{X^{(sc)}} \tag{3.10}$$

For example,

$$\langle X^2 \rangle_{P_X} = \langle X^2 \rangle_{P_{X^{(sc)}}} + \sigma_R^2 \tag{3.11}$$

which for the particular case considered gives the correct asymptotic limit

$$\lim_{t \to \infty} \langle X^2 \rangle_{P_X} = \frac{\gamma}{g} + \frac{\varepsilon}{2\gamma} \tag{3.12}$$

Dekker's systematic evaluation[19] is performed on the Master equation (corresponding to the Langevin equation) which reads as

$$\frac{\partial}{\partial t} P_X(X, t) = \left[-\frac{\partial}{\partial X} C(X) + \varepsilon \frac{\partial^2}{\partial X^2} \right] P_X(X, t), \qquad P_X(X, 0) = \delta(X) \tag{3.13}$$

This work, incidentally, is based on the integral transform method due to Haake.[13] Haake defines another distribution function

$$Q(X, t) = \int_{-\infty}^{\infty} du \frac{1}{[2\pi(\varepsilon/\gamma)]^{1/2}} \exp\left[-\frac{(X-u)^2}{2\varepsilon/\gamma} \right] P_X(u, t) \tag{3.14}$$

This integral transform implies the following time evolution for Q:

$$\frac{\partial Q}{\partial t}(X, t) = \left[-\frac{\partial}{\partial X} C\left(X + \frac{\varepsilon}{\gamma} \frac{\partial}{\partial X} \right) + \varepsilon \frac{\partial^2}{\partial X^2} \right] Q(X, t)$$

$$Q(X, 0) = \frac{1}{[2\pi(\varepsilon/\gamma)]^{1/2}} \exp\left[-\frac{X^2}{2\varepsilon/\gamma} \right] \tag{3.15}$$

The interesting feature of this transformation is that the initial distribution itself is smeared. Dekker's trick is to perform system size expansion[4] on the Master equation (3.15). Defining

$$X = Y(t) + \varepsilon^{1/2} R, \qquad \dot{Y} = C(Y), \qquad Y(t=0) = Y_0 \tag{3.16}$$

the irreducible solution of the Master equation can be obtained as

$$Q(X, t) = \left\langle \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left[-\frac{(X-Y)^2}{2\sigma^2} \right] \right\rangle_{Q_0},$$

$$Q_0 = \frac{1}{[2\pi(\varepsilon/\gamma)]^{1/2}} \exp\left[-\frac{Y_0^2}{2(\varepsilon/\gamma)} \right] \tag{3.17}$$

with

$$\dot{\sigma}^2 = 2C'(Y)(\sigma^2 - \varepsilon/\gamma) + 2\varepsilon, \qquad \sigma^2(0) = \varepsilon/\gamma \tag{3.18}$$

Now, the relation (3.14) implies

$$\langle X^n \rangle_{P_X} = \left\langle \left(\frac{\varepsilon}{2} \right)^{n/2} H_n\left(\frac{X}{(2\varepsilon)^{1/2}} \right) \right\rangle_Q \tag{3.19}$$

where $H_n(X)$ denote Hermite polynomial of nth order. Using Eqs. (3.16), (3.17), and (3.19) it is easy to show that

$$P_X(X, t) = \int_{-\infty}^{\infty} P_1(u, t)\, P_2(X-u, t)\, du \tag{3.20}$$

where

$$P_1(X, t) = \left(\frac{\partial \xi}{\partial X}\right) \frac{1}{[2\pi(\varepsilon/\gamma)]^{1/2}} \exp\left(\frac{-\xi^2}{2\varepsilon/\gamma}\right) \tag{3.21}$$

and

$$P_2(X, t) = \frac{1}{[2\pi(\sigma^2 - \varepsilon/\gamma)]^{1/2}} \exp\left[\frac{-X^2}{2(\sigma^2 - \varepsilon/\gamma)}\right] \tag{3.22}$$

Equation (3.21) can be identified with the scaling solution.[4] Similarly Eq. (3.22) is identical to the distribution function of the corrective remainder R introduced by Suzuki [see Eqs. (3.8) and (3.18)]. Thus we show the equivalence of both the methods.

The point we would like to stress is the following: Suzuki's *ad hoc* method of splitting the stochastic process into two statistically independent processes is systematically derived by Dekker. But, Dekker's method is applicable only for Gaussian white noise—since the Master equation itself cannot be explicitly written down for arbitrary noise, the integral transform method cannot be generalized to such processes. On the other hand, Suzuki's approximation is applied on the Langevin's equation itself and hence its generalization is straightforward.

3.2. Solution of Fokker–Planck Equation Using the Fer Formula

In order to explore the possibility of incorporating asymptotic fluctuations in a straightforward manner, we considered solving the Fokker–Planck equation in a different manner. The Fokker–Planck equation satisfied by the probability distribution of ξ reads as

$$\frac{\partial}{\partial t} P_\xi(\xi, t) = \varepsilon \hat{L}(\xi, t)\, P_\xi(\xi, t) \tag{3.23}$$

where

$$\hat{L}(\xi, t) = \frac{\partial}{\partial \xi} D(\xi, t) \frac{\partial}{\partial \xi} D(\xi, t), \qquad D(\xi, t) = \frac{C(\xi)}{C(X)} \tag{3.24}$$

[4] There is a small difference in the variance of ξ. Even this difference will vanish if one uses a time-dependent integral transformation, instead of the time-independent one used here.

The formal solution of Eq. (3.23) is given by

$$P_\xi(\xi, t) = T \exp\left[\varepsilon \int_0^t \hat{L}(\xi, t_1)\, dt_1\right] P_\xi(\xi, 0) \tag{3.25}$$

where T stands for time ordering. The above expression can systematically be expanded in ε using the Fer formula[22] as

$$P_\xi(\xi, t) = \prod_{n=1}^{\infty} \exp[\varepsilon^n \hat{M}_n(\xi, t)]\, P_\xi(\xi, 0) \tag{3.26}$$

where

$$\begin{aligned} \hat{M}_1 &= \int_0^t \hat{L}(\xi, t_1)\, dt_1 \\ \hat{M}_2 &= \int_0^t dt_2 \int_0^{t_2} dt_1 [\hat{L}(\xi, t_2)\, \hat{L}(\xi, t_1) - \hat{L}(\xi, t_1)\, \hat{L}(\xi, t_2)] \end{aligned} \tag{3.27}$$

etc. If $\hat{L}(\xi, t)$ is separable in ξ and t, then $\hat{M}_n = 0$ for $n \geqslant 2$. The first-order approximation is obtained as

$$P_\xi(\xi, t) = \exp[\varepsilon \hat{M}_1(\xi, t)]\, P_\xi(\xi, 0) \tag{3.28a}$$

Since this is a systematic expansion in ε alone, it is expected to be good for all times. It is easy to show that the unique solution of the operator $\hat{M}_1$ is given by

$$\hat{M}_1(\xi, t) = \frac{\partial}{\partial \xi} E(\xi, t) \frac{\partial}{\partial \xi} E(\xi, t), \qquad E^2 = \int_0^t D^2(\xi, t_1)\, dt_1 \tag{3.28b}$$

The solution of Eq. (3.28) becomes trivial if we define a new "stochastic process"

$$\omega = \int_0^\xi \frac{d\xi}{E(\xi, t)} \tag{3.29}$$

in a fictitious space of ε. Defining the probability distribution of ω in ε as $W(\omega, \varepsilon)$ and using the fact that $W(\omega, \varepsilon) = EP_\xi(\xi, t, \varepsilon)$, we get

$$\frac{\partial W(\omega, \varepsilon)}{\partial \varepsilon} = \frac{\partial^2 W(\omega, \varepsilon)}{\partial \omega^2}, \qquad W(\omega, \varepsilon = 0) = \delta(\omega) \tag{3.30}$$

The solution of Eq. (3.30) has to be done slightly carefully, owing to the

possibility that the range of ω need not be $[-\infty, \infty]$. Excepting for the trivial case of $C(X)=\gamma X$ the range $[-L, L)$ will be finite, where

$$L=\int_0^\infty \frac{d\xi}{E(\xi)} \tag{3.31}$$

The series solution of Eq. (3.30) is

$$W(\omega, \varepsilon)=\frac{1}{2L}\sum_{n=-\infty}^{\infty} \exp\left(\frac{-n^2\pi^2\varepsilon}{L^2}\right)\cos\frac{n\pi}{L}\omega \tag{3.32a}$$

which on Poisson transformation[27] becomes

$$W(\omega, \varepsilon)=\frac{1}{(4\pi\varepsilon)^{1/2}}\sum_{k=-\infty}^{\infty} \exp\left[-\frac{(\omega+2kL)^2}{4\varepsilon}\right] \tag{3.32b}$$

It can be shown that Eq. (3.32a) is the large time expansion and the other one is suitable for small times. In fact, retaining just one term in Eq. (3.32b) is good enough for times up to the scaling regime.

We have applied this to the model problem $C(X)=\gamma X-gX^3$, for which

$$\begin{aligned} E &= a_1\xi^6+a_2\xi^4+a_3\xi^2+a_4 \\ a_1 &= \frac{g^3}{4\gamma^4}[e^{4\gamma t}-6e^{2\gamma t}+12\gamma t+3+2e^{-2\gamma t}] \\ a_2 &= \frac{3g^2}{2\gamma^3}[e^{2\gamma t}-4\gamma t-e^{-2\gamma t}] \\ a_3 &= \frac{3g}{2\gamma^2}[2\gamma t-1+e^{-2\gamma t}] \\ a_4 &= \frac{1}{2\gamma}[1-e^{-2\gamma t}] \end{aligned} \tag{3.33}$$

The numerical calculations (not reported here) show that the present results are slightly better than the scaling results for times up to the scaling regime. Asymptotic results, however, are no better! We would now like to understand why this systematic expansion in ε of the Master equation has failed.

To make the argument clear we consider a one-dimensional model system $C(X)=\gamma X-gX^{2n+1}$ (n is a positive integer). Then from Eq. (2.19) we obtain

$$\langle X^{2m+1}\rangle=0, \qquad \langle X^{2m}\rangle=\left\langle\left[\frac{\xi e^{\gamma t}}{[1+(g/\gamma)\xi^{2n}(e^{2n\gamma t}-1)]^{1/2n}}\right]^{2m}\right\rangle_{P_\xi(\xi,t)} \tag{3.34}$$

where $P_\xi(\xi, t)$ is the exact probability distribution of ξ. It follows (from the definition of expectation values) that

$$\langle X^{2m} \rangle \leqslant \max \left\{ \frac{\xi e^{\gamma t}}{[1+(g/\gamma)\xi^{2n}(e^{2n\gamma t}-1)]^{1/2n}} \right\}^{2m} \tag{3.35}$$

irrespective of the probability distribution. Then we get

$$\lim_{t \to \infty} \langle X^{2m} \rangle \leqslant \left(\frac{\gamma}{g}\right)^{m/n}, \qquad \left(\frac{\gamma}{g}\right)^{1/n} = X_{\text{st}}^2 \tag{3.36}$$

where X_{st}^2 refers to the value of X^2 at the steady states. In fact the quantity in parenthesis in Eq. (3.34) is asymptotically equal to $(\gamma/g)^{m/n}$ for all ξ except at the point $\xi = 0$. Thus we see that independent of the probability distribution of ξ, $\langle X^{2m} \rangle$ asymptotically tend to $(\gamma/g)^{m/n}$. This in turn implies a δ function distribution in X[22]:

$$P_X(X, \infty) = |X_{\text{st}}| \, \delta(X^2 - X_{\text{st}}^2)$$

Thus the fluctuations cannot be asymptotically present in the transformed variable. The argument is applicable for all the cases where two stable minima exist.

The reason for this can be seen to be the noninvertibility of the transformation. For every ξ one and only one X exists. Therefore the transformation $\xi \to X$ is a one-to-one into mapping. However, the converse is not true. Asymptotically, only the X lying between the two stable steady states are in one-to-one correspondence with ξ [see Eq. (2.19)]. That is, there exists a region of the domain for which no image exists. This disqualifies the inverse transformation $X \to \xi$ to be a mapping.[28]

In view of this and on account of the great computational simplicity it offers, we believe the scaling theory is the best one can do in the scope of the particular nonlinear transformation.

4. ANALYSIS OF GAUSSIAN COLORED NOISE

The scaling formalism for the general case has been presented in Section 2. In this section we intend to assess the applicability of this method to handle Gaussian colored noise. For this purpose we consider the following one-dimensional model problem.

$$\dot{X} = X - X^3 + f(t) \tag{4.1}$$

$$\langle f(t) \rangle = 0, \qquad \langle f(t) f(t') \rangle = \varepsilon \Gamma \exp[-\Gamma |t - t'|] \tag{4.2}$$

where $f(t)$ is Gaussian. In the limit of $\Gamma = \infty$, this reduces to the white noise problem considered earlier. Then from Eq. (2.11) the scaling solution is

$$P_{\xi(\mathrm{sc})}(\xi, t) = \frac{1}{(2\pi\alpha)^{1/2}} \exp\left(\frac{-\xi^2}{2\alpha}\right) \tag{4.3}$$

where

$$\alpha = \begin{cases} \dfrac{\varepsilon\Gamma}{\Gamma^2 - 1}\left[\Gamma(1 - e^{-2t}) - (1 + e^{-2t} - 2e^{-(1+\Gamma)t}\right], & \text{for} \quad \Gamma \neq 1 \\ \dfrac{\varepsilon}{2}\left[1 - e^{-2t} - 2t\, e^{-2t}\right], & \text{for} \quad \Gamma = 1 \end{cases} \tag{4.4}$$

The various moments of the stochastic process in the scaling approximation read

$$\langle X^{2m+1} \rangle^{(\mathrm{sc})} = 0$$

$$\langle X^{2m} \rangle^{(\mathrm{sc})} = \frac{1}{(1 - e^{-2t})^m} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left(\frac{v^2\tau}{1 + v^2\tau}\right)^m e^{-v^2}\, dv$$

$$\tau = 2\alpha(e^{2t} - 1) \tag{4.5}$$

The switching time $T_s(\Gamma)$ is calculated as

$$T_s(\Gamma) = T_s(\Gamma = \infty) + \tfrac{1}{2} \ln(1 + 1/\Gamma) \tag{4.6}$$

Monte Carlo simulation of Eq. (4.1) was performed using a four point Runge–Kutta–Gill algorithm described earlier.[7,9] The exponentially correlated Gaussian noise is modeled as the Ornstein–Uhlenbeck process

$$\dot{f} = \Gamma[-f + \eta(t)], \qquad P_f(f, 0) = \frac{1}{(2\pi\varepsilon\Gamma)^{1/2}} \exp\left(\frac{-f^2}{2\varepsilon\Gamma}\right) \tag{4.7}$$

where $\eta(t)$ is Gaussian and white with the normalization

$$\langle \eta(t) \rangle = 0, \qquad \langle \eta(t)\, \eta(t') \rangle = 2\varepsilon\delta(t - t') \tag{4.8}$$

The computations were performed with $\varepsilon = 0.5 \times 10^{-6}$. The values of Γ used in the simulation were ∞, 1, 0.1, and 0.01 so as to cover very short and large correlation times, respectively. 2500 realizations of the process were found to give good statistics. The second moment $\langle X^2 \rangle$ as a function of time is presented in Fig. 3 for various Γ. Scaling results are also shown for comparison. Figure 4 gives similar information about the fourth

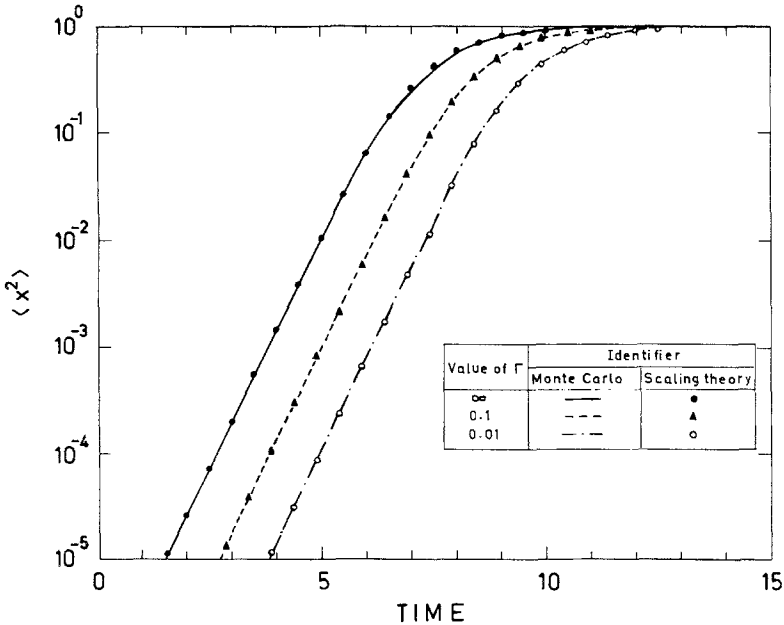

Fig. 3. The second moment $\langle X^2 \rangle$ of the stochastic process $\dot{X} = X - X^3 + f(t)$ with $P_X(X, 0) = \delta(X)$. Here, the $f(t)$ is modeled to be Gaussian with $\langle f(t) \rangle = 0$ and $\langle f(t) f(t') \rangle = 5 \times 10^{-7} \Gamma \exp[-\Gamma|t - t'|]$. The results obtained from the Monte Carlo simulation are compared with the corresponding scaling results for $\Gamma = \infty$, 0.1, and 0.01. The onset of order is delayed when the correlation time Γ^{-1} increases.

moment. In fact, we have presented $(\langle X^4 \rangle - \langle X^2 \rangle^2)^{1/2}$ which is much more sensitive to the fluctuations. The agreement is very good. We have also verified that the switching time T_s follows precisely the scaling relation (4.6).

From the point of view of physics, the finite correlation time does not introduce any qualitative difference in the fluctuation behavior (at least in the quantities considered). Quantitatively, the onset of macroscopic order is delayed when the correlation time increases.[5]

Before closing this section, we could like to make a few remarks. First, we have not computed unequal time correlations and compared them with the corresponding scaling result. We believe the agreement should be good in that case also, when it is carried out. Second (and more important), we have performed the Monte Carlo simulation with Gaussian noise only. In view of the central limit theorem, the intrinsic fluctuations are most likely to be Gaussian. Hence the present formulation may be of use in analyzing

[5] If we keep $\varepsilon\Gamma$ (the strength of the colored noise) constant and vary Γ, then the opposite situation arises. This, of course, cannot lead to the white noise limit.

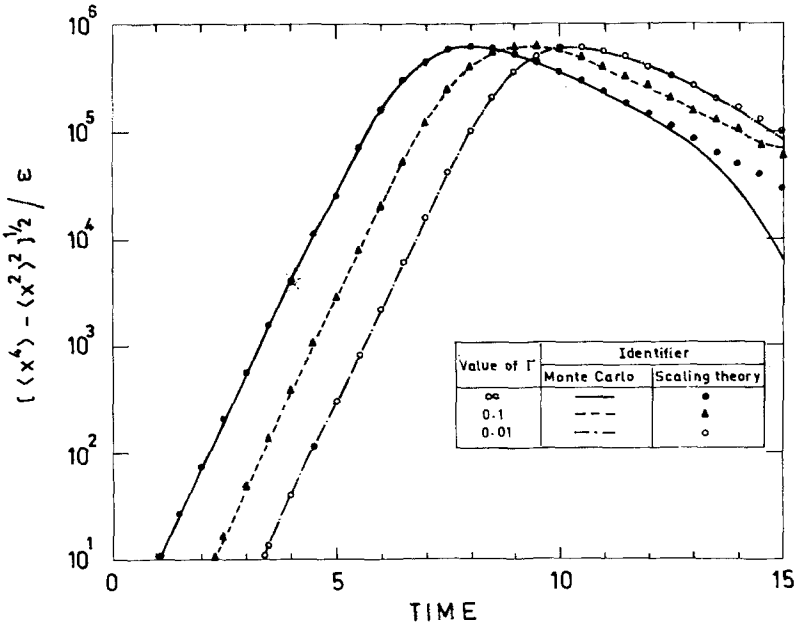

Fig. 4. The normalized standard deviation $(\langle X^4\rangle - \langle X^2\rangle^2)^{1/2}/\varepsilon$ of the process X^2 for the problem considered in Fig. 3. The magnitude of the fluctuation enhancement can be seen to be independent of the correlation time.

most of the physical problems where intrinsic fluctuations are important. At the same time we must mention that the method should be able to tackle other distributions too, which may occur when the system is driven externally. In such situations Monte Carlo simulations with non-Gaussian noise are essential for the judgement.

5. CONCLUSIONS

The present investigation clearly establishes the usefulness of the scaling theory in unraveling fluctuation phenomena occurring in coupled physical systems driven by Gaussian colored noise. The scaling theory, together with the saddle point approximation, provides a simple but elegant means for getting closed form expressions for the various moments of the stochastic process. The nonlinear transformation used in arriving at the scaling result is noninvertible and hence the unified treatment is essential for preserving asymptotic fluctuations.

The scaling approximation does not subsume any particular statistical property of the driving noise. Hence the present mathematical formalism

should be applicable for treating any general colored noise. Such a situation may arise when the system is externally driven.

Before concluding, it is tempting to speculate that a similar analysis should be possible for handling stochastic nonlinear fields, as well.

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